MATH3560 **History of Mathematics** **Session 1, 2009**

Theory: Quaternions

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> Quaternions... are like the elephant's trunk, ready at *any* moment for *anything*, be it to pick up a crumb or a field gun, to strangle a tiger, or to uproot a tree. Portable in the extreme, applicable everywhere... directed by a little native who requires no special skill or training, and who can be transferred from one elephant to another without much hesitation.
>
> – Peter Tait to Arthur Cayley (1894 CE)[(2)]

William Rowan Hamilton needed to understand everything - for him, "the design of physical science is... to learn the language and to interpret the oracles of the universe."[(1)]

Born in Dublin in 1805 CE[(6)], as a child he quickly learned to speak several languages (some sources say as few as seven languages[(6)], others as many as thirteen[(2)]), and foreign languages continued to be important to him throughout his life. They allowed him to read mathematics published from all over Europe at a time when mathematics was developing at a explosive rate, and although his 887 page book *Lectures on Quaternions* was written primarily in English, it contains passages written in French, German and Latin, sometimes with no English translation provided to readers.

The story of his Quaternions begins with a quadratic equation:

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}$$

[1]Elephant sketch taken from: www.taveepong.com

It was known since at least 1545 CE[2] that depending on the values given for a, b and c, the solutions for x might be 'real' or 'imaginary'. For example, the classic equation:

$$x^2 + 1 = 0$$

has solution $x = \sqrt{-1}$, which was called an imaginary number since it did not sit on the number line extending from $(-\infty, \infty) = \mathbb{R}$.

The mystery of imaginary numbers took the next few hundred years for mathematicians to unravel. In the 1830s (CE), Hamilton began his own attempt to better understand them, inspired by the work of Jean-Robert Argand in visualising complex numbers[3]. Hamilton was not happy with the idea that solutions to quadratic equations 'might' be of different forms depending on the given coefficients in an equation. Hamilton tried to formalise exactly what sort of object the solution to a quadratic equation was, irrespective of the coefficients.

Hamilton began (very *abstractly*, a term he himself used) by considering what he called *couples*:

$$(a, b), \quad a, b \in \mathbb{R}$$

Hamilton then tried to develop an 'algebra' for couples – he wanted to find ways to add, subtract, multiply and divide couples, in analogous ways to the regular algebra for real numbers. Hamilton was also interested in maintaining the associative, commutative and distributive laws of algebra (and also used those exact terms.)

Hamilton ended up developing the following rules for couples:

$$\begin{aligned}
(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2) && \text{i.e. Addition} \\
(a_1, b_1) - (a_2, b_2) &= (a_1 - a_2, b_1 - b_2) && \text{i.e. Subtraction} \\
(a_1, b_1) \times (a_2, b_2) &= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) && \text{i.e. Multiplication} \\
(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) \div (a_1, b_1) &= (a_2, b_2) && \text{i.e. Division} \\
\lambda \times (a_1, b_1) &= (\lambda \times a_1, \lambda \times b_1) && \text{i.e. Scalar multiplication} \\
(\lambda \times a_1, \lambda \times b_1) \div (a_1, b_1) &= \lambda && \text{i.e. Scalar division} \\
(a_1, 0) &= a_1 && \text{i.e. Degeneracy, cf. } z = x + i0 = x
\end{aligned}$$

These rules meant that the associative, commutative and distributive laws of algebra were maintained, for example:

$$\begin{aligned}
(a_1, b_1) \times ((a_2, b_2) \times (a_3, b_3)) &= (a_1, b_1) \times (a_2a_3 - b_2b_3, a_2b_3 + a_3b_2) \\
&= (a_1(a_2a_3 - b_2b_3) - b_1(a_2b_3 + a_3b_2), a_1(a_2b_3 + a_3b_2) + b_1(a_2a_3 - b_2b_3)) \\
&= (a_1a_2a_3 - a_1b_2b_3 - b_1a_2b_3 - b_1a_3b_2, a_1a_2b_3 + a_1a_3b_2 + b_1a_2a_3 - b_1b_2b_3) \\
&= (a_1a_2a_3 - b_1a_3b_2 - a_1b_2b_3 - b_1a_2b_3, a_1a_2b_3 - b_1b_2b_3 + a_1a_3b_2 + b_1a_2a_3) \\
&= (a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + b_1a_2), b_3(a_1a_2 - b_1b_2) + a_3(a_1b_2 + b_1a_2)) \\
&= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) \times (a_3, b_3) \\
&= ((a_1, b_1) \times (a_2, b_2)) \times (a_3, b_3) \qquad \text{i.e. Associative law of multiplication}
\end{aligned}$$

One of the most important results of Hamilton's exploration of couples was that he could now express:

$$(0,1) \times (0,1) = (-1,0) = -1, \quad \text{i.e. } (0,1)^2 = -1$$

For Hamilton, this removed some of the mystery of quadratic equations. It meant that solutions could always be interpreted as couples, which he now felt he had a thorough understanding of, and "all notion of anything *imaginary, unreal,* or *impossible* [was] excluded from... view".[(3)]

Hamilton's couples form a field, and are equivalent to the field $\mathbb{C}$ of complex numbers, usually written in the form $a + ib, a, b \in \mathbb{R}, i = \sqrt{-1}$. In the case of Hamilton's couples, he had $i = (0,1)$. Hamilton was the first to show this equivalence between complex numbers and an algebra of couples[(1)].

Note that the term *field* was not used as a mathematical definition until many years after the development of couples (Richard Dedekind first introduced the notion in 1871 CE.[(5)]) However at the same time that Hamilton was playing around with couples, his lifelong friend Augustus De Morgan was busy trying to find a minimum set of axioms needed to form an algebra, so was essentially working on the problem of defining a field.[(3)]

After his success with couples, Hamilton then wondered whether he could create an algebra for triplets,

$$(a, b, c), \quad a, b, c \in \mathbb{R}$$

Hamilton wanted to "do for the analysis of three-dimensional space what imaginary numbers do for two-dimensional space."[(2)]

Before discussing the problems he had with triplets, first a remark on notation: at some point during Hamilton's attempt to develop triplet algebra (perhaps after being exposed to the i, j notation of Charles Graves[(3)]), he began writing his (a, b, c) triplets as $(a + bi + cj)$, where $a = (a, 0, 0) = a \times (1, 0, 0) = a \times 1$ as in the degeneracy law for couples, $i = (0, 1, 0)$ and $j = (0, 0, 1)$. In this way $1, i, j$ formed a basis for triplets.

One of the problems that Hamilton recurrently ran into when developing triplet algebra was that, it seemed no matter what rules were used to define the algebra, Hamilton could always find situations where two non-zero triplets (a_1, b_1, c_1) and (a_2, b_2, c_2), when multiplied together, would give a zero triplet. Consider:

$$\begin{aligned}(a_1, b_1, c_1) \times (a_2, b_2, c_2) &= (a_1 + b_1 i + c_1 j) \times (a_2 + b_2 i + c_2 j) \\ &= a_1 a_2 + a_1 b_2 i + a_1 c_2 j + b_1 i a_2 + b_1 i b_2 i + b_1 i c_2 j + c_1 j a_2 + c_1 j b_2 i + c_1 j c_2 j\end{aligned}$$

(in all attempts at his triplet algebra, Hamilton wanted a distributive law like above to hold)

$$= a_1 a_2 + (a_1 b_2 + b_1 a_2) i + (a_1 c_2 + c_1 a_2) j + b_1 b_2 i^2 + b_1 c_2 ij + c_1 b_2 ji + c_1 c_2 j^2$$

At this point, Hamilton had to decide what i^2, j^2, ij and ji actually meant in the triplet algebra. In the couple case, he had $i^2 = (0, 1) \times (0, 1) = -1$. Therefore Hamilton would often assume that $i^2 = -1$ for the triplet case too, and similarly assume that $j^2 = -1$. But then he was still left trying to decide what to do with ij and ji. Were they equal? If they were equal, did commutativity hold in general in the triplet algebra, or was this a special case? For this example, let's assume Hamilton tried $ij = ji = 1$ along with his typical $i^2 = j^2 = -1$, then:

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (a_1a_2 - b_1b_2 - c_1c_2 + b_1c_2 + c_1b_2) + (a_1b_2 + b_1a_2)i + (a_1c_2j + c_1a_2)j$$

And if we plug in the values $(a_1, b_1, c_1) = (0, 1, 1) = (a_2, b_2, c_2)$ then:

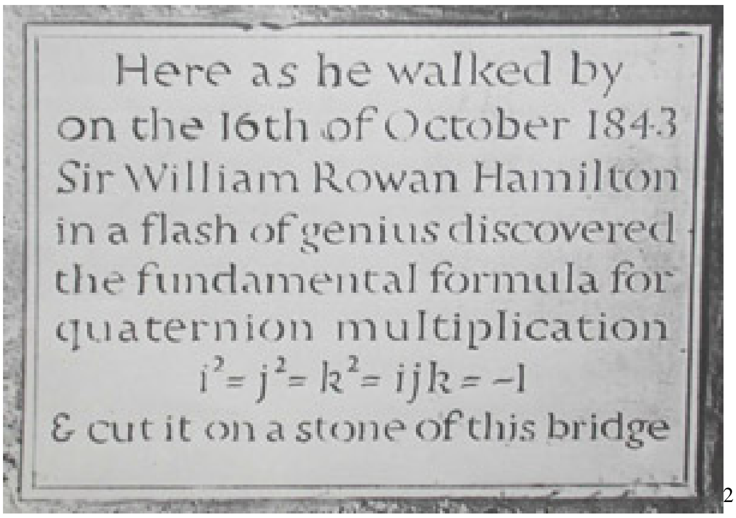
$$\begin{aligned}(0, 1, 1) \times (0, 1, 1) &= (0 - 1 - 1 + 1 + 1) + (0)i + (0)j \\ &= 0\end{aligned}$$

This annoyed Hamilton greatly. He did not want the product of two non-zero triplets to equal zero$^{(3)}$, as that does not happen in our usual fields $\mathbb{R}$ or $\mathbb{C}$. For example if we have $xy = 0$ for $x, y \in \mathbb{R}$, then we are guaranteed that either $x = 0$, $y = 0$, or $x = y = 0$. Not so in triplets.

Another annoyance for Hamilton was that he could not find a way for the *law of moduli* to work in triplets, as it had done in $\mathbb{R}$ and $\mathbb{C}$. For example, given any two numbers $x, y \in \mathbb{R}$, we have $||x \times y|| = ||x|| \times ||y||$, where $||x|| = \sqrt{x^2} = |x|$ is the usual norm in $\mathbb{R}$.

Similarly in $\mathbb{C}$, given any two couples (a_1, b_1) and (a_2, b_2) we have $||(a_1, b_1) \times (a_2, b_2)|| = ||(a_1, b_1)|| \times ||(a_2, b_2)||$, where $||(a, b)|| = \sqrt{a^2 + b^2}$ is the usual norm in $\mathbb{C}$. For triplets, Hamilton wanted a law like $||(a_1, b_1, c_1) \times (a_2, b_2, c_2)|| = ||(a_1, b_1, c_1)|| \times ||(a_2, b_2, c_2)||$ to hold, but he was unable to find an acceptable algebra for triplets where it would.

The breakthrough for Hamilton came when he decided to give up on creating an algebra for triples and focus on an algebra for quadruples instead. Sneak preview of what's to come:



[2]

[2]Photo taken from: http://plus.maths.org

Immediately prior to his breakthrough, Hamilton had been toying around with setting $i^2 = -1$, $j^2 = -1$, and $ij = k = -ij$, where k was some yet undefined constant like 1 or -1. As an aside, Hamilton was the first person in history to consider using non-commutative products[(1)], and probably inspired other mathematicians to (conscientiously!) break the traditional rules of algebra in order to develop new ideas.[(2)]

Using these laws Hamilton calculated again the product of two general triplets:

$$\begin{aligned}(a_1, b_1, c_1) \times (a_2, b_2, c_2) &= (a_1 + b_1 i + c_1 j) \times (a_2 + b_2 i + c_2 j) \\ &= a_1 a_2 + (a_1 b_2 + b_1 a_2)i + (a_1 c_2 + c_1 a_2)j + b_1 b_2 i^2 + b_1 c_2 ij + c_1 b_2 ji + c_1 c_2 j^2 \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + b_1 a_2)i + (a_1 c_2 + c_1 a_2)j + (b_1 c_2 - c_1 b_2)k\end{aligned}$$

This was a problem for Hamilton. He wanted the product of two triplets to be another triplet, but it wasn't clear that $(a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + b_1 a_2)i + (a_1 c_2 + c_1 a_2)j + (b_1 c_2 - c_1 b_2)k$ could actually be expressed as a triplet i.e. in the form (a_3, b_3, c_3).

Putting that aside for a moment, Hamilton decided to check whether the *law of moduli* had been satisfied in this case. For triplets, we define $||(a, b, c)|| = \sqrt{a^2 + b^2 + c^2}$. Hamilton wanted:

$$||(a_1, b_1, c_1)|| \times ||(a_2, b_2, c_2)|| = ||(a_1, b_1, c_1) \times (a_2, b_2, c_2)||$$

i.e., assuming we can write $(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (a_3, b_3, c_3)$, then he needed:

$$\begin{aligned}\sqrt{a_1^2 + b_1^2 + c_1^2} \times \sqrt{a_2^2 + b_2^2 + c_2^2} &= ||(a_1, b_1, c_1) \times (a_2, b_2, c_2)|| \\ &= \sqrt{a_3^2 + b_3^2 + c_3^2}\end{aligned}$$

which is true when:

$$(a_1^2 + b_1^2 + c_1^2) \times (a_2^2 + b_2^2 + c_2^2) = (a_3^2 + b_3^2 + c_3^2) \tag{1}$$

Hamilton then noticed that the following (general purpose) identity was always true:

$$(a_1^2 + b_1^2 + c_1^2) \times (a_2^2 + b_2^2 + c_2^2) = \left(\begin{array}{c} (a_1 a_2 - b_1 b_2 - c_1 c_2)^2 + (a_1 b_2 + b_1 a_2)^2 \\ +(a_1 c_2 + c_1 a_2)^2 + (b_1 c_2 - c_1 b_2)^2 \end{array} \right) \tag{2}$$

Hamilton wondered whether it was possible to reduce the sum of four squares shown in (2) to a sum of only three squares as seen in (1). Crucially he noted that the earlier product of two triplets had really given him a *quadruple* over $1, i, j, k$:

$$\begin{aligned}(a_1, b_1, c_1) \times (a_2, b_2, c_2) &= (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + b_1 a_2)i + (a_1 c_2 + c_1 a_2)j + (b_1 c_2 - c_1 b_2)k \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2, \quad a_1 b_2 + b_1 a_2, \quad a_1 c_2 + c_1 a_2, \quad b_1 c_2 - c_1 b_2)\end{aligned}$$

And so, considering the quadruple and the expression for a sum of four squares appearing in (2), Hamilton realised that his triplets were really just a degenerate case of what he named Quaternions, that is, quadruples (a, b, c, d) over basis elements $1, i, j, k$, with $d = 0$ for triplets.

Hamilton then proceeded to lock down the laws of quaternions. He kept his assumptions that $i^2 = j^2 = -1$ and $ij = k = -ji$, but now needed to calculate values for ik, ki, jk, kj and k^2. Since $ij \neq ji$, Hamilton assumed that ik and ki, and jk and kj were not necessarily commutative either. However, he assumed the algebra did maintain associativity and found:

$$\begin{aligned}
ik &= i(k) = i(ij) = (ii)j = -j \\
ki &= (k)i = (-ji)i = -j(ii) = j && (= -ik) \\
jk &= j(k) = j(-ji) = -(jj)i = i \\
kj &= (k)j = (ij)j = i(jj) = -i && (= -jk) \\
k^2 &= (k)(k) = (ij)(-ji) = -i(jj)i = ii = -1
\end{aligned}$$

Thus, on the 16th of October, 1843 CE[6], Hamilton attained the fundamental law for quaternion algebra, carved in stone and then photographed and reprinted in this essay,[3]

$$i^2 = j^2 = k^2 = ijk = -1$$

All in all, Hamilton had spent over thirteen years[2] on the journey from couples to triplets to quaternions. After discovering quaternions he proceeded to spend the remainder of his life exploring their applications, writing over 100 papers.[2] In the same style as the fields $\mathbb{R}$ and $\mathbb{C}$, Quaternions are now commonly referred to as the field $\mathbb{H}$ in honour of Hamilton. In fact $\mathbb{H}$ is actually a skew-field, which is just like a regular field except we no longer guarantee commutativity for multiplication.

Hamilton's friend John Graves quickly extended the idea of quaternions to form the octaves (or octonions) in December 1843 CE[6]. The octonions were (no prizes for guessing!) octuplets of the form (a, b, c, d, e, f, g, h) over eight basis elements. However, unlike quaternions the octonions did not maintain the associative law of algebra. In 1898 CE Adolf Hurwitz proved that there were no possible further extensions[6], but not before Arthur Cayley wasted many hours trying to find a 16-tuple algebra.

It is worth pointing out here that Hamilton actually coined the now familiar terms *scalar* and *vector* during his discovery of quaternions. His original definitions:

- "Scalars"... are simply those positives and negatives, on the scale of progression from $-\infty$ to $+\infty$, which are commonly called *reals* (or real quantities) in algebra.[3]
- "the vector of the point B, from the point A:" because it may be considered as having for its office, function, work, task, or business, to *transport* or CARRY (in Latin, *vehere*) a *moveable point*, from the given or initial position A, to the sought or final position B.[3]

Hamilton also invented[1] what we now call the *dot product* and *cross product* of two vectors, although he called them the *scalar product* and *vector product*, respectively. Hamilton thought of them as just components of the larger quaternion product, i.e. the multiplication of quadruples. This similarity in lingo means that by modern standards Hamilton's original lectures are still quite readable, although he is *extremely* wordy.

[3] Actually the photograph is of a commemorative plaque, since Hamilton's original carving faded away.

By the early 20th century[2], quaternion algebra as Hamilton taught it had been largely abandoned in favour of modern (then and still now) vector analysis. In a large part, vector analysis was born out of a painful triage process performed on quaternions - the pain here most often felt by ardent followers of quaternions. For example Peter Tait, a former student of Hamilton, described one of the first major works on vector analysis as "a sort of hermaphrodite monster... of the notions of Hamilton"[1]

Quaternions as Hamilton taught them are still used in very specific applications, namely for handling three dimensional rotations, where they outshine other methods.[1] As Andrew Hanson writes, "there is a relationship between quaternions and three-dimensional rotations that permits the three rotational degrees of freedom to be represented exactly be the three degrees of freedom of a normalised quaternion."[4] In particular, the use of quaternions helps avoid rotational disaster situations known as gimbal-locks.[7]

It makes sense that quaternions might be good at encoding 3D rotations, since quaternions are an extension of complex numbers - which we know are good at encoding 2D rotations. Unfortunately, further discussion of this relationship is prevented by the strict word limit on this essay, which I am already well beyond.

As a final treat, it is worth noting that Simon Altmann and Michael Crowe write that both Carl Gauss[2;1] and Olinde Rodrigues[1] independently discovered equivalent forms of quaternions prior to Hamilton.

References

[1] Simon L. Altmann. Hamilton, Rodrigues, and the Quaternion Scandal. *Mathematics Magazine*, 62(5):291–308, December 1989.

[2] Michael Crowe. *A History of Vector Analysis*. University of Notre Dame Press, Notre Dame, Indiana, 1967. (I read a seventeen page summary of the book prepared by the author, but could not locate a copy of the original book).

[3] William Rowan Hamilton. *Lectures on Quaternions*. Hodges and Smith, Dublin, Ireland, 1853.

[4] Andrew J. Hanson. *Visualizing Quaternions*. Morgan Kaufmann Publishers, San Francisco, California, 2006.

[5] Karen Hunger Parshall Jeremy Gray. *Episodes in the history of modern algebra (1800-1950)*. AMS Bookstore, 2007.

[6] Bartel Leendert van der Waerden. Hamilton's Discovery of Quaternions. *Mathematics Magazine*, 49(5):227–234, November 1976.

[7] Sobeit Void. Quaternion Powers. http://www.gamedev.net/reference/articles/article1095.asp, February 2003. Last accessed: 5th May, 2009.