MATH5785 **Geometry** **Session 1, 2009**

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Contents

Problem 1

For the triangle defined below, find the:

1. Midpoints of its sides
2. Feet of its altitudes
3. Midpoints of its orthocenter/point segments (i.e. midpoints of the lines joining the orthocenter O with the points A_1, A_2 and A_3)

And show that the 9-point circle goes through all these points.

Solution

Finite field $\mathbb{F}_7$, where $7 = 0$.

Given three points forming a triangle:

$$\begin{aligned} A_1 &= [x_1, y_1] = [0, 0] \\ A_2 &= [x_2, y_2] = [2, 3] \\ A_3 &= [x_3, y_3] = [-1, 1] \end{aligned}$$

With corresponding quadrances:

$$\begin{aligned} Q_1 = Q(A_2, A_3) &= (2 - -1)^2 + (3 - 1)^2 = 3^2 + 2^2 = 9 + 4 = 13 = 7 + 6 = 0 + 6 = 6 \\ &= -1 \\ Q_2 = Q(A_1, A_3) &= (0 - -1)^2 + (0 - 1)^2 = 1 + 1 \\ &= 2 \\ Q_3 = Q(A_1, A_2) &= (0 - 2)^2 + (0 - 3)^2 = 2^2 + 3^2 = 4 + 9 + 13 \\ &= -1 \end{aligned}$$

Consider the lines:

$$\begin{aligned} A_1A_2 &=< y_1 - y_2 : x_2 - x_1 : x_1y_2 - x_2y_1 > \\ &=< 0 - 3 : 2 - 0 : 0 > \\ &=< -3 : 2 : 0 > \\ &\equiv -3x + 2y = 0 \qquad (= l_3) \\ A_1A_3 &=< y_1 - y_3 : x_3 - x_1 : x_1y_3 - x_3y_1 > \\ &=< 0 - 1 : -1 - 0 : 0 > \\ &=< -1 : -1 : 0 > \end{aligned}$$

$$
\begin{aligned}
&\equiv -x-y=0 && (=l_2)\\
A_2A_3 &=< y_2-y_3 : x_3-x_2 : x_2y_3-x_3y_2 > \\
&=< 3-1 : -1-2 : (2)(1)-(-1)(3) > \\
&=< 2 : -3 : 5 > \\
&\equiv 2x-3y+5=0 && (=l_1)
\end{aligned}
$$

And their corresponding spreads:

$$
\begin{aligned}
s_1 = s(l_2,l_3) &= \frac{(a_2b_3-a_3b_2)^2}{(a_2^2+b_2^2)(a_3^2+b_3^2)}\\
&= \frac{((-1)(2)-(-3)(-1))^2}{((-1)^2+(-1)^2)((-3)^2+2^2)}\\
&= \frac{(-5)^2}{(2)(13)}\\
&= \frac{25}{26} = \frac{21+4}{21+5} = \frac{4}{5} = \frac{12}{15} = \frac{12}{14+1} = \frac{12}{1} = 12 = 7+5\\
&= 5\\
s_2 = s(l_1,l_3) &= \frac{(a_1b_3-a_3b_1)^2}{(a_1^2+b_1^2)(a_3^2+b_3^2)}\\
&= \frac{((2)(2)-(-3)(-3))^2}{(2^2+(-3)^2)(2^2+(-3)^2)}\\
&= \frac{(-5)^2}{(4+9)(4+9}\\
&= \frac{25}{169} = \frac{21+4}{168+1} = \frac{4}{1}\\
&= 4\\
s_3 = s(l_1,l_2) &= \frac{(a_1b_2-a_2b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)}\\
&= \frac{((2)(-1)-(-1)(-3))^2}{(2^2+(-3)^2)((-1)^2+(-1)^2)}\\
&= \frac{25}{(4+9)(1+1)} = \frac{25}{26} = 5
\end{aligned}
$$

Quick check of the spread law:

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$$
$$\Longleftrightarrow$$
$$\frac{5}{-1} = \frac{4}{2} = \frac{5}{-1} \equiv -5 = 2, \text{ true in } \mathbb{F}_7$$

The midpoints of the sides of the triangle are:

$$\begin{aligned}
M_1 = M(A_2, A_3) &= \left[\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right] \\
&= \left[\frac{1}{2}, \frac{4}{2}\right] \\
&= [4, 2] \\
M_2 = M(A_1, A_3) &= \left[\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right] \\
&= \left[\frac{-1}{2}, \frac{1}{2}\right] \\
&= [-4, 4] \\
M_3 = M(A_1, A_2) &= \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] \\
&= \left[1, \frac{3}{2}\right] = [1, 12] \\
&= [1, 5]
\end{aligned}$$

The three lines l_1, l_2, l_3 are each non-null lines.

Given a point $A = [x, y]$ and a non-null line l, the altitude n from A to l meets l at:

$$F = nl = \left[\frac{b^2x - aby - ac}{a^2 + b^2}, \frac{a^2y - abx - bc}{a^2 + b^2}\right]$$

So the feet of the altitudes of the triangle are:

$$\begin{aligned}
F_1 = F(A_1, l_1) &= \left[\frac{b^2x - aby - ac}{a^2+b^2}, \frac{a^2y - abx - bc}{a^2+b^2}\right] \\
&= \left[\frac{-(2)(5)}{(2)^2+(-3)^2}, \frac{-(-3)(5)}{(2)^2+(-3)^2}\right] \\
&= \left[\frac{-10}{13}, \frac{15}{13}\right] = \left[\frac{4}{6}, \frac{1}{6}\right] = \left[\frac{24}{36}, \frac{6}{36}\right] \\
&= [3, 6] \\
F_2 = F(A_2, l_2) &= \left[\frac{(-1)^2(2) - (-1)(-1)(3)}{(-1)^2+(-1)^2}, \frac{(-1)^2(3) - (-1)(-1)(2)}{(-1)^2+(-1)^2}\right] \\
&= \left[\frac{-1}{2}, \frac{1}{2}\right] \\
&= [-4, 4] \\
F_3 = F(A_3, l_3) &= \left[\frac{2^2(-1) - (-3)(2)}{13}, \frac{(-3)^2(1) - (-3)(2)(-1)}{13}\right] \\
&= \left[\frac{2}{13}, \frac{3}{13}\right] = \left[\frac{2}{6}, \frac{3}{6}\right] = [12, 18] \\
&= [5, 4]
\end{aligned}$$

Altitude line: For any point $A = [x, y]$, and line $l =< a : b : c >$, there is a unique line n *perpendicular* to l, passing through A.

$$n =< -b : a : bx - ay >$$

Therefore, the altitude lines are:

$$\begin{aligned}
a_1 = a(A_1, l_1) &=< 3 : 2 : 0 > \\
a_2 = a(A_2, l_2) &=< 1 : -1 : (-1)(2) - (-1)(3) > \\
&=< 1 : -1 : 1 > \\
a_3 = a(A_3, l_3) &=< -2 : -3 : (2)(-1) - (-3)(1) > \\
&=< -2 : -3 : 1 >
\end{aligned}$$

The orthocenter O of the triangle is given by the intersection of its altitude lines.

$$3x + 2y = 0 \quad (1)$$

$$x - y + 1 = 0 \quad (2)$$

$$-2x - 3y + 1 = 0 \quad (3)$$

Now, $(2) \implies x = y - 1$, so therefore $(1) \implies 3(y-1) + 2y = 0 \implies 3y - 3 + 2y = 0 \implies 5y - 3 = 0 \implies 5y = 3 \implies y = \frac{3}{5} = \frac{9}{15} = 2$, so $x = y - 1 = 2 - 1 = 1$.

Therefore $O = [1, 2]$.

We need to look at the midpoints of the lines between O and each of A_1, A_2, A_3:

$$Q_1 = M(A_1, O) = \left[\frac{1}{2}, \frac{2}{2}\right] = [4, 1]$$
$$Q_2 = M(A_2, O) = \left[\frac{3}{2}, \frac{5}{2}\right] = [12, 20] = [5, 6]$$
$$Q_3 = M(A_3, O) = \left[0, \frac{3}{2}\right] = [0, 5]$$

Now, we have 9 points and we need to check they all lie on the same circle:

$$M_1 = [4, 2]$$
$$M_2 = [-4, 4]$$
$$M_3 = [1, 5]$$
$$F_1 = [3, 6]$$
$$F_2 = [-4, 4]$$
$$F_3 = [5, 4]$$
$$Q_1 = [4, 1]$$
$$Q_2 = [5, 6]$$
$$Q_3 = [0, 5]$$

Circles are defined by:

$$(x - a)^2 + (y - b)^2 = K, \text{ for some constant } K$$

We have:

From M1: $(4 - a)^2 + (2 - b)^2 = (4 - a)(4 - a) + (2 - b)(2 - b) = 16 - 8a + a^2 + 4 - 4b + b^2$
$= 20 - 8a - 4b + a^2 + b^2 = K$

From M2: $(-4 - a)^2 + (4 - b)^2 = (-4 - a)(-4 - a) + (4 - b)(4 - b) = 16 + 8a + a^2 + 16 - 8b + b^2$
$= 32 + 8a - 8b + a^2 + b^2 = K$

From M3: $(1 - a)^2 + (5 - b)^2 = (1 - a)(1 - a) + (5 - b)(5 - b) = 1 - 2a + a^2 + 25 - 10b + b^2$
$= 26 - 2a - 10b + a^2 + b^2 = K$

These agree when:

$$20 - 8a - 4b = 32 + 8a - 8b = 26 - 2a - 10b \implies 0 = 12 + 16a - 4b = 6 + 6a - 6b$$
$$\implies a = b - 1, \text{ and so } 0 = 12 + 16(b - 1) - 4b \implies 0 = 12 + 16b - 16 - 4b \implies 4 = 12b$$

So $b = \frac{4}{12} = \frac{4}{5} = 12 = 5$ in $\mathbb{F}_7$. So $a = b - 1 = 5 - 1 = 4$, and our circle is defined by:

$$(x - 4)^2 + (y - 5)^2 = K$$

Substituting in $M_3 = [1, 5]$ gives $K = 9$, which agrees with all 9 points.

Problem 2

Given

$$S_2(\alpha) = \beta, S_2(\beta) = \alpha$$

Find all solutions to this pair of quadratic equations. One is:

$$\alpha = \frac{5 - \sqrt{5}}{8}, \beta = \frac{5 + \sqrt{5}}{8}$$

Solution

Second Spread Polynomial: (Definition)

$$S_2(s) = 4s(1 - s)$$

$$S_2(\alpha) = 4\alpha(1 - \alpha) = 4\alpha - 4\alpha^2 = \beta \qquad (1)$$
$$S_2(\beta) = 4\beta(1 - \beta) = 4\beta - 4\beta^2 = \alpha \qquad (2)$$

Substituting (2) into (1) we have:

$$\begin{aligned} 4(4\beta - 4\beta^2) - 4(4\beta - 4\beta^2)^2 &= \beta \\ 16\beta - 16\beta^2 - 4(4\beta - 4\beta^2)(4\beta - 4\beta^2) - \beta &= 0 \\ 15\beta - 16\beta^2 - 4(16\beta^2 - 32\beta^3 + 16\beta^4) &= 0 \\ 15\beta - 16\beta^2 - 64\beta^2 + 128\beta^3 - 64\beta^4 &= 0 \\ 15\beta - 80\beta^2 + 128\beta^3 - 64\beta^4 &= 0 \end{aligned}$$

One obvious solution at this point is $\beta = 0$, and then $\alpha = 4\beta - 4\beta^2 = 0$. Otherwise, continuing on we have:

$$15 - 80\beta + 128\beta^2 - 64\beta^3 = 0$$

Now, we are told that both $(\beta - \frac{5-\sqrt{5}}{8})$ and $(\beta - \frac{5+\sqrt{5}}{8})$ are factors of this, which means that $(\beta - \frac{5-\sqrt{5}}{8})(\beta - \frac{5+\sqrt{5}}{8}) = 64\beta^2 - 80\beta + 20$ is a factor, which gives us (after some polynomial division):

$$(\frac{3}{4} - \beta)(64\beta^2 - 80\beta + 20) = 0$$

So another solution is $\beta = \frac{3}{4} = \alpha$.[1]

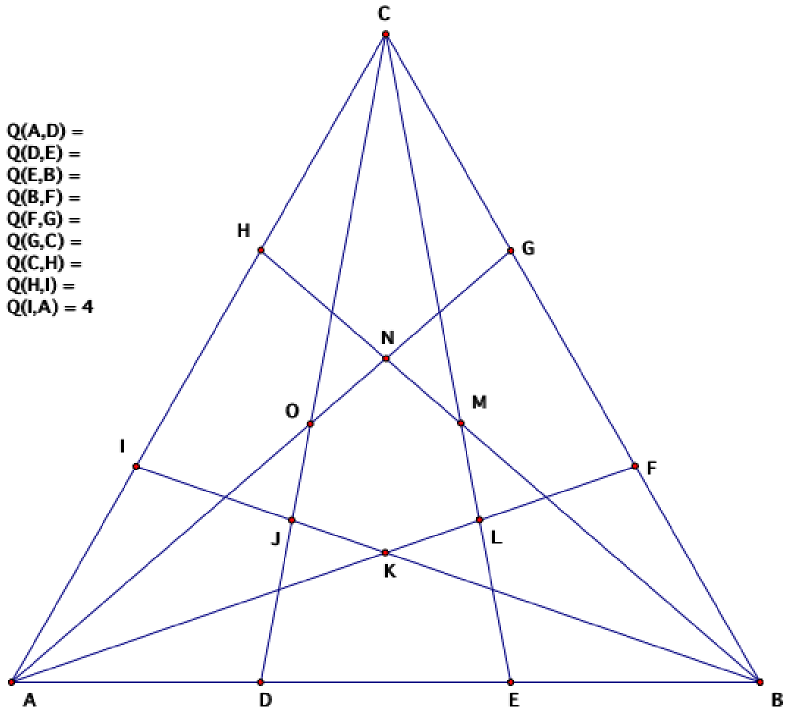
Alternate proof: Just assume $\alpha = \beta$, then:

$$4s(1 - s) = s \implies 4(1 - s) = 1 \implies 1 - s = \frac{1}{4} \implies s = \frac{3}{4}$$

[1] Also shown at: "WildTrig25: Pentagons and five-fold symmetry"

Problem 3

Given an equilateral triangle.
Divide each side into three equal pieces ("sides trisected.")
Find as many quadrances as possible.
Find the spreads of the interior hexagon.



Solution

Equilateral triangle has angles of 60 degrees, so spreads of $\frac{3}{4}$.

Firstly, $Q(A,E) = Q(B,D) = Q(B,G) = Q(C,F) = Q(C,I) = Q(A,H) = 16$.

The cross law tells us that:

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3)$$
$$(Q_1 + Q_2 - Q_3) = \pm\sqrt{4Q_1Q_2(1 - s_3)}$$
$$Q_3 = Q_1 + Q_2 \pm \sqrt{4Q_1Q_2(1 - s_3)}$$

So if we consider consider $\overline{ABF}$, where $Q(A,B) = 36$, we have:

$$\begin{aligned} Q_3 &= Q_1 + Q_2 \pm \sqrt{4Q_1Q_2(1-s_3)} \\ &= 4 + 36 \pm \sqrt{\frac{576}{4}} = 40 \pm 12 \end{aligned}$$

Since $Q(A,B) = 36 > Q(A,F)$, we have $Q(A,F) = 40 - 12 = 28$.

Now we can work out s_A and s_F in $\overline{ABF}$ using the spread law:

$$\begin{aligned} \frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} &\implies \frac{s_A}{4} = \frac{s_F}{36} = \frac{\frac{3}{4}}{28} = \frac{3}{112} \\ &\implies s_A = \frac{12}{112} = \frac{3}{28}, \text{ and } s_F = \frac{108}{112} = \frac{27}{28} \end{aligned}$$

Similarly in $\overline{ACG}$, we have $s_A = \frac{3}{28}$ and $s_G = \frac{27}{28}$, and $Q(A,G) = 28$.

Similarly in $\overline{ABI}$, we have $s_B = \frac{3}{28}$ and $s_I = \frac{27}{28}$, and $Q(B,I) = 28$.

Similarly in $\overline{BCH}$, we have $s_B = \frac{3}{28}$ and $s_H = \frac{27}{28}$, and $Q(B,H) = 28$.

Similarly in $\overline{ACD}$, we have $s_C = \frac{3}{28}$ and $s_D = \frac{27}{28}$, and $Q(A,D) = 28$.

Similarly in $\overline{BCE}$, we have $s_C = \frac{3}{28}$ and $s_E = \frac{27}{28}$, and $Q(C,E) = 28$.

This lets us find s_A in $\overline{AFG}$ using the cross law:

$$\begin{aligned} (Q(A,F) + Q(A,G) - A(G,F))^2 &= 4Q(A,F)Q(A,G)(1-s_A) \\ (28 + 28 - 4)^2 &= 4(28)(28)(1-s_A) \\ 2704 - 3136 &= -3136 s_A \\ \frac{27}{196} &= s_A \end{aligned}$$

Similarly $s_B = \frac{27}{196}$ in $\overline{BHI}$

Similarly $s_C = \frac{27}{196}$ in $\overline{CDE}$

We can also find s_J in $\overline{BDJ}$ using the triple spread formula:

$$(s_D + s_B + s_J)^2 = 2(s_D^2 + s_B^2 + s_J^2) + 4s_Ds_Bs_J$$
$$(\frac{27}{28} + \frac{3}{28} + s_J)^2 = 2((\frac{27}{28})^2 + (\frac{3}{28})^2 + s_J^2) + 4(\frac{27}{28})(\frac{3}{28})s_J$$
$$(\frac{30}{28} + s_J)^2 = 2((\frac{729}{784}) + (\frac{9}{784}) + s_J^2) + 4(\frac{81}{784})s_J$$
$$(\frac{30}{28} + s_J)(\frac{30}{28} + s_J) = \frac{1476}{784} + 2s_J^2 + 4(\frac{81}{784})s_J$$
$$\frac{900}{784} + \frac{60}{28}s_J + s_J^2 = \frac{1476}{784} + 2s_J^2 + 4(\frac{81}{784})s_J$$
$$\frac{900}{784} - \frac{1476}{784} = s_J^2 - \frac{60}{28}s_J + \frac{324}{784}s_J$$
$$s_J^2 - \frac{1356}{784}s_J + \frac{576}{784} = 0$$

So, the quadratic formula tells us that:

$$\begin{aligned}
s_J &= \frac{\frac{1356}{784} \pm \sqrt{(\frac{1356}{784})^2 - \frac{2304}{784}}}{2} \\
&= \frac{\frac{1356}{784} \pm \sqrt{\frac{1838736}{614656} - \frac{1806336}{614656}}}{2} \\
&= \frac{\frac{1356}{784} \pm \sqrt{\frac{32400}{614656}}}{2} \\
&= \frac{\frac{1356}{784} \pm \frac{180}{784}}{2} \\
&= \frac{\frac{1356}{784} + \frac{180}{784}}{2} \text{ or } \frac{\frac{1356}{784} - \frac{180}{784}}{2} \\
&= \frac{768}{784} \text{ or } \frac{588}{784} \\
&= \frac{768}{784} = \frac{48}{49}
\end{aligned}$$

Therefore $s_J = s_L = s_N = \frac{48}{49}$

This lets us work out $Q(D, J)$ in $\overline{BDJ}$ using the spread law:

$$\frac{s_B}{Q(D,J)} = \frac{s_J}{Q(D,B)} \implies \frac{\frac{3}{28}}{Q(D,J)} = \frac{48}{784}$$
$$\implies \frac{Q(D,J)}{\frac{3}{28}} = \frac{784}{48}$$
$$\implies Q(D,J) = \frac{784}{48} \times \frac{3}{28} = \frac{2352}{1344} = \frac{147}{84} = \frac{49}{28} = \frac{7}{4}$$

So $Q(D,J) = Q(E,L) = Q(F,L) = Q(G,N) = Q(H,N) = Q(I,J) = \frac{7}{4}$

Now, let us turn to $\overline{ACO}$. We can use the triple spread formula to find s_O here:

$$\begin{aligned}
(\frac{3}{28} + \frac{3}{28} + s_O)^2 &= 2(2(\frac{3}{28})^2 + s_O^2) + 4(\frac{3}{28})^2 s_O \\
(\frac{6}{28} + s_O)(\frac{6}{28} + s_O) &= \frac{36}{784} + 2s_O^2 + \frac{36}{784}s_O \\
\frac{36}{784} + \frac{12}{28}s_O + s_O^2 &= \frac{36}{784} + 2s_O^2 + \frac{36}{784}s_O \\
0 &= s_O^2 - \frac{300}{784}s_O
\end{aligned}$$

So,

$$s_O = \frac{300}{784} = \frac{75}{196}$$

Therefore $s_O = s_K = s_M = \frac{75}{196}$

So the spreads of the interior hexagon are:

$$\begin{aligned}
s_J = s_L = s_N &= \frac{48}{49} \\
s_O = s_K = s_M &= \frac{75}{196}
\end{aligned}$$

Problem 4

Create a test example and verify this works.
Work in 3D space, imagine y-axis is the train track.
Pick a point for the flagpole e.g. $[-5, 3, 8]$
Know H, can verify the formula (it should pop out as one of the solutions.)

Solution

For $B = [-5, 3, 8]$, we know that $H = 64$ and $A_3 = [-5, 3, 0]$.

We want to verify that this is a solution of our quadratic equation for H in terms of r_1, r_2, r_3 and P_1, P_2.

$$P_2(H(\frac{1}{r_3} - \frac{1}{r_1}) + P_1)^2 = P_1(H(\frac{1}{r_3} - \frac{1}{r_2}) + P_2)^2$$

Pick $A_1 = [0, 0, 0]$, $A_2 = [0, 10, 0]$, $D = [0, 5, 0]$.

Then $P_1 = Q(A_1, D) = 25$

And $P_2 = Q(A_2, D) = 25$.

And $P_3 = Q(A_3, D) = 29$.

We also know that $Q_2 = Q(A_1, A_3) = 25 + 9 = 34$.

And $Q_1 = Q(A_2, A_3) = 25 + (3 - 10)^2 = 25 + 49 = 74$.

So $r_1 = \frac{H}{H+Q_2} = \frac{64}{64+34} = \frac{64}{98} = \frac{32}{49}$

And $r_2 = \frac{H}{H+Q_1} = \frac{64}{64+74} = \frac{64}{138} = \frac{32}{69}$.

And $r_3 = \frac{H}{H+P_3} = \frac{64}{64+29} = \frac{64}{93}$.

So, need to verify:

$$P_2(H(\frac{1}{r_3} - \frac{1}{r_1}) + P_1)^2 = P_1(H(\frac{1}{r_3} - \frac{1}{r_2}) + P_2)^2$$
$$(25)(H(\frac{1}{\frac{64}{93}} - \frac{1}{\frac{32}{49}}) + 25)^2 = (25)(H(\frac{1}{\frac{64}{93}} - \frac{1}{\frac{32}{69}}) + 25)^2$$
$$(25)(H(\frac{93}{64} - \frac{49}{32}) + 25)^2 = (25)(H(\frac{93}{64} - \frac{69}{32}) + 25)^2$$

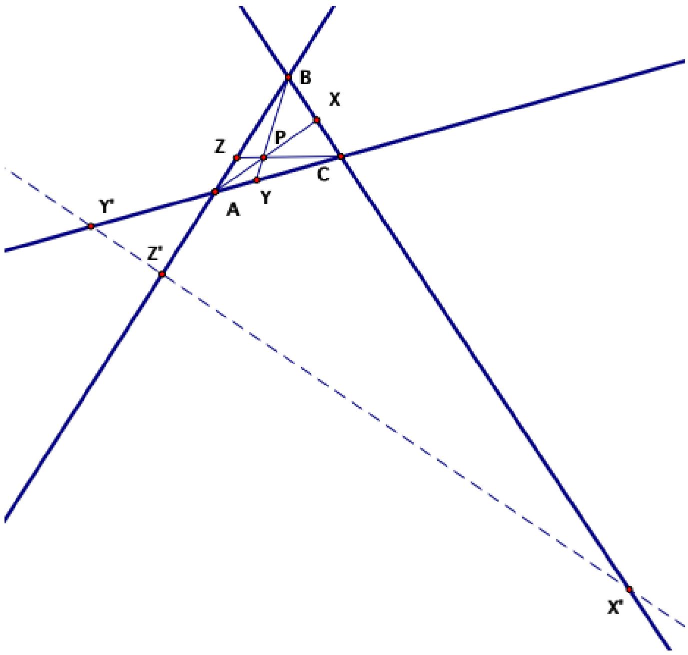
$$
\begin{aligned}
(25)(H(\frac{93}{64}-\frac{98}{64})+25)^2 &= (25)(H(\frac{93}{64}-\frac{138}{64})+25)^2 \\
(H(\frac{93}{64}-\frac{98}{64})+25)^2 &= (H(\frac{93}{64}-\frac{138}{64})+25)^2 \\
(64(\frac{93}{64}-\frac{98}{64})+25)^2 &= (64(\frac{93}{64}-\frac{138}{64})+25)^2 \\
(93-98+25)^2 &= (93-138+25)^2 \\
(20)^2 &= (-20)^2
\end{aligned}
$$

Which is true, so $H = 64$ is a solution.

Problem 5

Given $H(BC, XX')$ ("on BC, X and X' are harmonic conjugates".)
Given $H(AB, ZZ')$ ("on AB, Z and Z' are harmonic conjugates".)
Given $H(AC, YY')$.

Show X', Y', Z' are collinear.
Lines in triangle are "random lines".
Chosen some point P (arbitrary).



Solution

Given $Y = xA + yC$, we have $Y' = xA - yC$.

Similarly given $Z = xA + qB$, we have $Z' = xA - qB$.

Similarly given $X = rB + yC$, we have $X' = rB - yC$.

Now $Y' - X' = xA - yC - (rB - yC) = xA - rB$.

So the lines $X'Y'$ and AB intersect at the point $P_1 = xA - rB$.

Therefore the point $Z' = xA - qB$ could lie on the line $Y'X'$, if we can show that $q = r$.

Now $Z + yC = xA + qB + yC$

And $Y + qB = xA + yC + qB$

So the lines ZC and YB intersect at the point $xA + qB + yC$

Similarly $X + xA = rB + yC + xA$

And $Y + rB = xA + yC + rB$

So the lines XA and YB intersect at the point $xA + rB + yC$

But (by construction) we know the three lines ZC, YB and XA all intersect at the same point P.

So $P = xA + qB + yC = xA + rB + yC$ and hence $q = r$.

So $Z' = xA - qB = xA - rB = P_1$ lies on the line $X'Y'$, and hence X', Y', Z' are collinear.

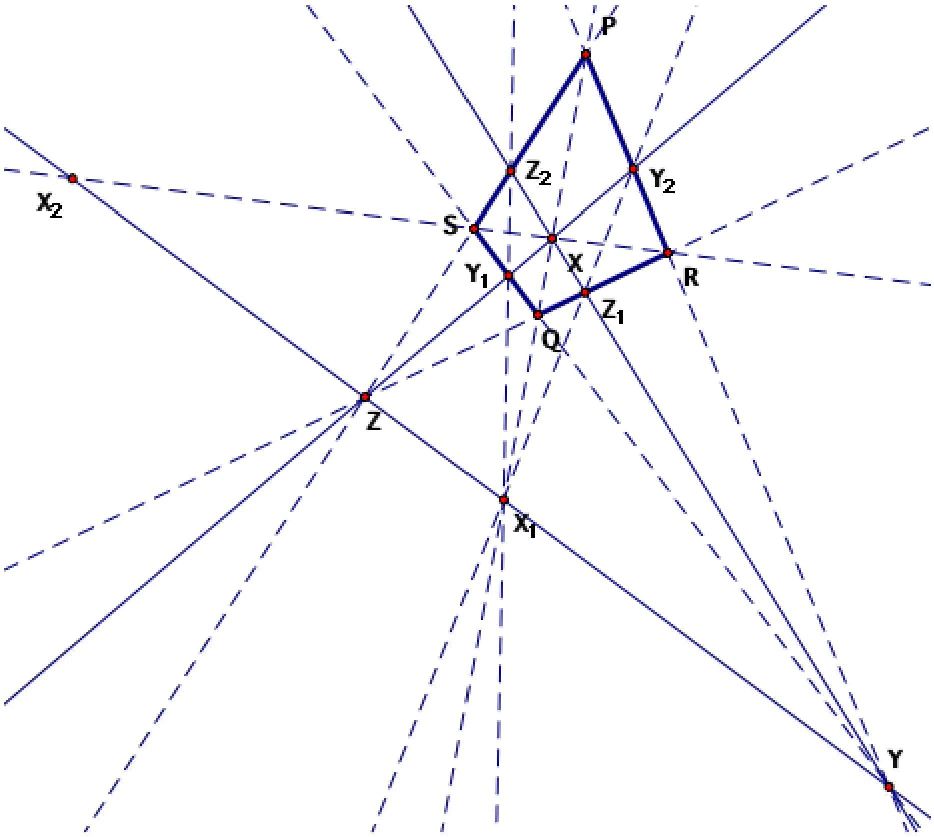
Problem 6

$\overline{XYZ}$ diagonal triangle.

$H(YZ, X_1X_2)$,
$H(XY, Z_1Z_2)$,
$H(XZ, Y_1Y_2)$ all harmonic ranges.

Also many harmonic pencils e.g. ZZ_2, ZX, ZZ_1, ZY.

Show that $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are 6 points of a complete quadrilateral.

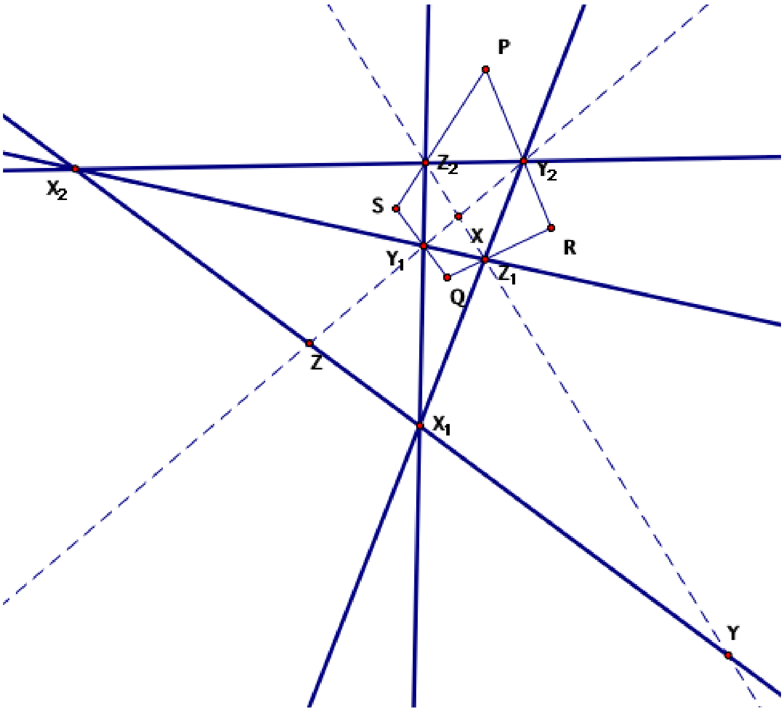


Solution

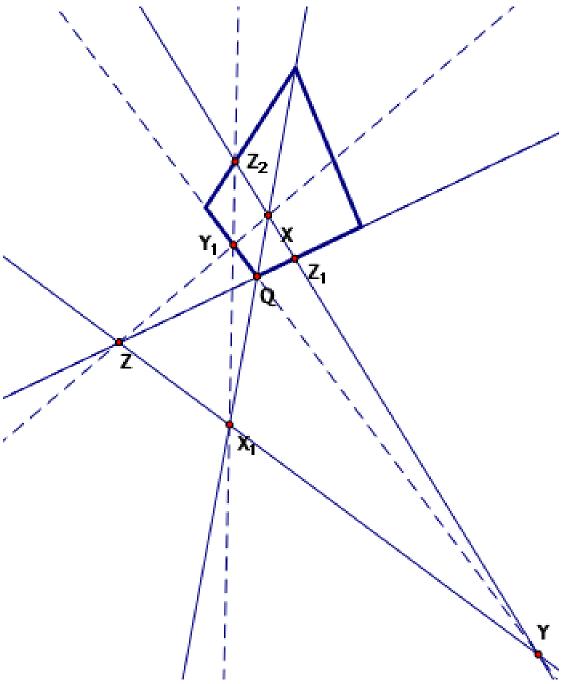
Complete quadrilateral: (Definition)

The figure determined by four lines, no three of which are concurrent, and their six points of intersection. A complete quadrilateral has three diagonals. The midpoints of the diagonals of a complete quadrilateral are collinear.

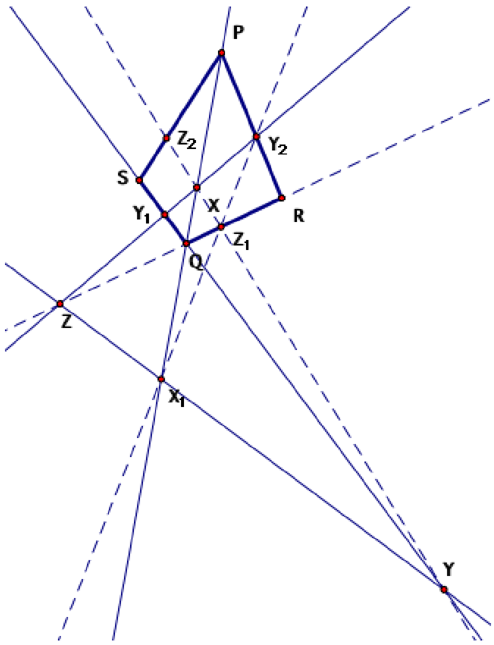
Picture "proof" of the complete quadrilateral to start with:



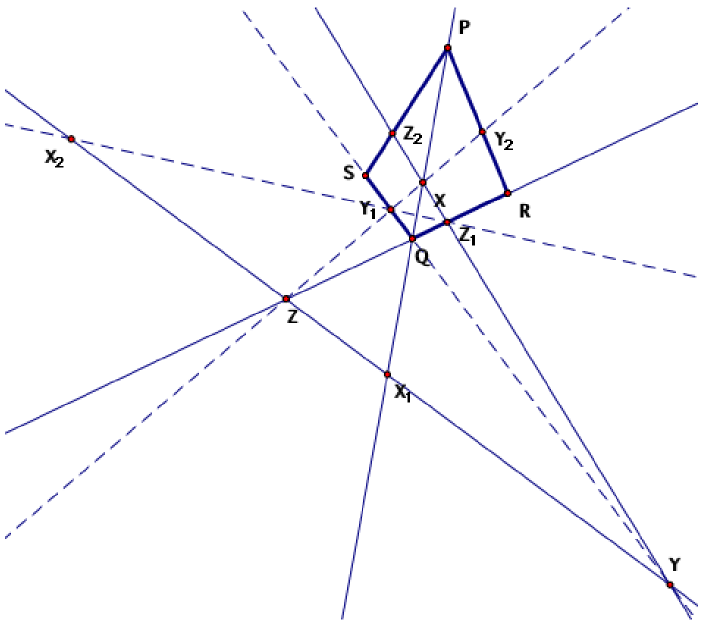
Why does this work? Well, we have X_1, Y_1, Z_2 collinear by construction as that's how Z_2 was found for $H(XY, Z_1Z_2)$ (note, you may need to rotate your head to the left)



And why are X_1, Y_2, Z_1 collinear? Well, that's also true by construction, since that's how Y_2 was found for $H(XZ, Y_1Y_2)$ (more head rotation, this time to the right!)



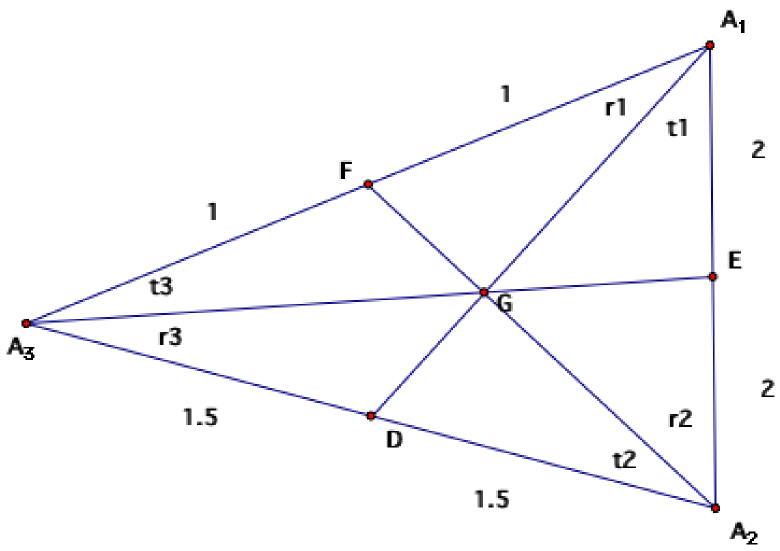
And why are X_2, Y_1, Z_1 collinear? Well, that's how X_2 can be found for $H(YZ, X_1X_2)$ (no head rotation necessary!)



And why are X_2, Y_2, Z_2 collinear? Well, from Problem 5 we already proved that X_2, Y_2, Z_2 are collinear (considering X_1, Y_1 and Z_1 on $\overline{XYZ}$.)

Problem 7

Calculate the spreads $r_1, t_1, r_2, t_2, r_3, t_3$.
The numbers are distances.
The lines are medians.



Solution

We have:

$$Q(A_1, A_2) = 4^2 = 16$$
$$Q(A_1, A_3) = 2^2 = 4$$
$$Q(A_2, A_3) = 3^2 = 9$$

The median quadrance theorem tells us that:

$$Q(A_3, E) = \frac{Q(A_1, A_3) + Q(A_2, A_3)}{2} - \frac{Q(A_1, A_2)}{4}$$
$$= \frac{4+9}{2} - \frac{16}{4} = \frac{13}{2} - \frac{8}{2} = \frac{5}{2}$$

and similarly,

$$Q(A_1, D) = \frac{Q(A_1, A_3) + Q(A_1, A_2)}{2} - \frac{Q(A_2, A_3)}{4}$$
$$= \frac{4+16}{2} - \frac{9}{4} = \frac{40}{4} - \frac{9}{4} = \frac{31}{4}$$

$$\begin{aligned}Q(A_2,F)&=\frac{Q(A_1,A_2)+Q(A_2,A_3)}{2}-\frac{Q(A_1,A_3)}{4}\\&=\frac{16+9}{2}-\frac{4}{4}=\frac{25}{2}-\frac{2}{2}=\frac{23}{2}\end{aligned}$$

The spread law tells us that:

$$\frac{t_3}{Q(A_1,E)}=\frac{s_{A_1}}{Q(A_3,E)}\implies\frac{t_3}{4}=\frac{s_{A_1}}{\frac{5}{2}}\implies t_3=\frac{8s_{A_1}}{5}$$

and similarly,

$$\begin{aligned}\frac{r_3}{Q(A_2,E)}&=\frac{s_{A_2}}{Q(A_3,E)}\implies\frac{r_3}{4}=\frac{s_{A_2}}{\frac{5}{2}}\implies r_3=\frac{8s_{A_2}}{5}\\\frac{t_2}{Q(A_3,F)}&=\frac{s_{A_3}}{Q(A_2,F)}\implies t_2=\frac{s_{A_3}}{\frac{23}{2}}\implies t_2=\frac{2s_{A_3}}{23}\\\frac{r_2}{Q(A_1,F)}&=\frac{s_{A_1}}{Q(A_2,F)}\implies r_2=\frac{s_{A_1}}{\frac{23}{2}}\implies r_2=\frac{2s_{A_1}}{23}\\\frac{t_1}{Q(A_2,D)}&=\frac{s_{A_2}}{Q(A_1,D)}\implies\frac{t_1}{\frac{9}{4}}=\frac{s_{A_2}}{\frac{31}{4}}\implies t_1=\frac{9s_{A_2}}{31}\\\frac{r_1}{Q(A_3,D)}&=\frac{s_{A_3}}{Q(A_1,D)}\implies\frac{r_1}{\frac{9}{4}}=\frac{s_{A_3}}{\frac{31}{4}}\implies r_1=\frac{9s_{A_3}}{31}\end{aligned}$$

This tells us that:

$$\begin{aligned}\frac{31}{9}t_1=s_{A_2}=\frac{5}{8}r_3&\implies t_1=\frac{9}{31}\times\frac{5}{8}r_3=\frac{45}{248}r_3\\\frac{23}{2}t_2=s_{A_3}=\frac{31}{9}r_1&\implies t_2=\frac{2}{23}\times\frac{31}{9}r_1=\frac{62}{207}r_1\\\frac{5}{8}t_3=s_{A_1}=\frac{23}{2}r_2&\implies t_3=\frac{8}{5}\times\frac{23}{2}r_2=\frac{184}{10}r_2=\frac{92}{5}r_2\end{aligned}$$

Quick double check: the laws of proportion for triangles tells us that:

$$\frac{r_1}{r_2}=\frac{s_1}{s_2}\frac{R_1}{R_2}$$

and here we have medians with $R_1=R_2$, so,

$$\begin{aligned}\frac{t_3}{r_3}=\frac{s_{A_1}}{s_{A_2}}&\implies\frac{\frac{92}{5}r_2}{r_3}=\frac{\frac{23}{2}r_2}{\frac{5}{8}r_3}\implies\frac{92}{5}=\frac{23}{2}\times\frac{8}{5}=\frac{92}{5}\\\frac{t_1}{r_1}=\frac{s_{A_2}}{s_{A_3}}&\implies\frac{\frac{45}{248}r_3}{r_1}=\frac{\frac{5}{8}r_3}{\frac{31}{9}r_1}\implies\frac{45}{248}=\frac{5}{8}\times\frac{9}{31}=\frac{45}{248}\\\frac{t_2}{r_2}=\frac{s_{A_3}}{s_{A_1}}&\implies\frac{\frac{62}{207}r_1}{r_2}=\frac{\frac{31}{9}r_1}{\frac{23}{2}r_2}\implies\frac{62}{207}=\frac{31}{9}\times\frac{2}{23}=\frac{62}{207}\end{aligned}$$

Now, we still need to find values for r_1, r_2, r_3. Let's relate them to the quadrances we know.

In $\overline{A_2A_3E}$, the spread law tells us that:

$$\frac{s_E}{9} = \frac{r_3}{4} = \frac{s_{A_2}}{\frac{5}{2}} \implies \frac{s_E}{9} = \frac{r_3}{4} = \frac{\frac{5}{8}r_3}{\frac{5}{2}}$$
$$\implies s_E = \frac{9}{4}r_3$$

Similarly, in $\overline{A_1A_3E}$, we have:

$$\frac{s_{A_1}}{\frac{5}{2}} = \frac{t_3}{4} = \frac{s_E}{4} \implies \frac{\frac{23}{2}r_2}{\frac{5}{2}} = \frac{\frac{92}{5}r_2}{4} = \frac{s_E}{4}$$
$$\implies s_E = \frac{92}{5}r_2$$

So $\frac{92}{5}r_2 = \frac{9}{4}r_3$

In $\overline{A_1A_3D}$, the spread law tells us that:

$$\frac{r_1}{\frac{9}{4}} = \frac{s_D}{4} = \frac{s_{A_3}}{\frac{31}{4}} \implies \frac{r_1}{\frac{9}{4}} = \frac{s_D}{4} = \frac{\frac{31}{9}r_1}{\frac{31}{4}}$$
$$\implies s_D = \frac{16}{9}r_1$$

And similarly in $\overline{A_1A_2D}$, we have:

$$\frac{s_D}{16} = \frac{t_1}{\frac{9}{4}} = \frac{s_{A_2}}{\frac{31}{4}} \implies \frac{s_D}{16} = \frac{\frac{45}{248}r_3}{\frac{9}{4}} = \frac{\frac{5}{8}r_3}{\frac{31}{4}}$$
$$\implies s_D = \frac{16\frac{5}{8}r_3}{\frac{31}{4}} = \frac{40}{31}r_3$$

So $\frac{16}{9}r_1 = \frac{40}{31}r_3$

In $\overline{A_2A_3F}$, the spread law tells us that:

$$t2 = \frac{s_{A_3}}{\frac{23}{2}} = \frac{s_F}{9} \implies \frac{62}{207}r_1 = \frac{\frac{31}{9}r_1}{\frac{23}{2}} = \frac{s_F}{9}$$
$$\implies s_F = \frac{558}{207}r_1 = \frac{62}{23}r_1$$

And similarly in $\overline{A_1A_2F}$, we have:

$$r_2 = \frac{s_{A_1}}{\frac{23}{2}} = \frac{s_F}{16} \implies r_2 = \frac{\frac{23}{2}r_2}{\frac{23}{2}} = \frac{s_F}{16}$$
$$\implies s_F = 16r_2$$

And so $\frac{62}{23}r_1 = 16r_2$

So, now we know that:

$$\frac{92}{5}r_2 = \frac{9}{4}r_3$$
$$\frac{16}{9}r_1 = \frac{40}{31}r_3$$
$$\frac{62}{23}r_1 = 16r_2$$

Now, **behold the cross law:**

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3)$$

We can use the cross law to find S_{A_1}, S_{A_2} and S_{A_3}:

$$(4 + 16 - 9)^2 = 4(4)(16)(1 - s_{A_1}) \implies 121 = 256 - 256s_{A_1}$$
$$\implies s_{A_1} = \frac{135}{256}$$

And similarly,

$$(9 + 16 - 4)^2 = 4(9)(16)(1 - s_{A_2}) \implies 441 = 576 - 576s_{A_2}$$
$$\implies s_{A_2} = \frac{15}{64}$$
$$(9 + 4 - 16)^2 = 4(9)(4)(1 - s_{A_3}) \implies 9 = 144 - 144s_{A_3}$$
$$\implies s_{A_3} = \frac{15}{16}$$

Therefore:

$$r_2 = \frac{s_{A_1}}{\frac{23}{2}} = \frac{\frac{135}{256}}{\frac{23}{2}} = \frac{135}{256} \times \frac{2}{23} = \frac{270}{5888} = \frac{135}{2944}$$

And hence,

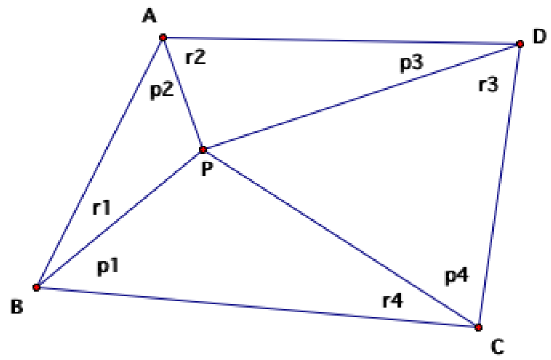
$$r_1 = 16 \times \frac{23}{62} \times \frac{135}{2944} = \frac{49680}{182528} = \frac{3105}{11408} = \frac{135}{496}$$
$$r_3 = \frac{92}{5} \times \frac{4}{9} \times \frac{135}{2944} = \frac{49680}{132480} = \frac{3105}{8280} = \frac{621}{1656} = \frac{207}{552} = \frac{69}{184} = \frac{3}{8}$$

Subbing these values into our formulas for t_1, t_2, t_3 gives:

$$t_1 = \frac{45}{248} r_3 = \frac{45}{248} \times \frac{69}{184} = \frac{3105}{45632} = \frac{135}{1984}$$
$$t_2 = \frac{62}{207} r_1 = \frac{62}{207} \times \frac{3105}{11408} = \frac{192510}{2361456} = \frac{96255}{1180728} = \frac{10695}{131192} = \frac{15}{184}$$
$$t_3 = \frac{92}{5} r_2 = \frac{92}{5} \times \frac{135}{2944} = \frac{12420}{14720} = \frac{3105}{3680} = \frac{621}{736} = \frac{27}{32}$$

Problem 8

Random point P.
Show $r_1r_2r_3r_4 = p_1p_2p_3p_4$.



Solution

Some fun with the spread law:

$$\frac{r_1}{Q(A,P)} = \frac{p_2}{Q(B,P)}$$
$$\frac{r_2}{Q(D,P)} = \frac{p_3}{Q(A,P)}$$
$$\frac{r_3}{Q(C,P)} = \frac{p_4}{Q(D,P)}$$
$$\frac{r_4}{Q(B,P)} = \frac{p_1}{Q(C,P)}$$

Therefore,

$$\frac{r_1}{Q(A,P)} \times \frac{r_2}{Q(D,P)} \times \frac{r_3}{Q(C,P)} \times \frac{r_4}{Q(B,P)} = \frac{p_1}{Q(C,P)} \times \frac{p_2}{Q(B,P)} \times \frac{p_3}{Q(A,P)} \times \frac{p_4}{Q(D,P)}$$
$$\frac{r_1r_2r_3r_4}{Q(A,P)Q(B,P)Q(C,P)Q(D,P)} = \frac{p_1p_2p_3p_4}{Q(A,P)Q(B,P)Q(C,P)Q(D,P)}$$
$$r_1r_2r_3r_4 = p_1p_2p_3p_4$$

Problem 9

(AB, CD) is the cross ratio of four points A, B, C, D.

If $(AB, CD) = k$, prove that:

1. $(AB, DC) = \frac{1}{k}$
2. $(AC, DB) = \frac{1}{1-k}$
3. $(AD, BC) = \frac{k-1}{k}$

Solution

Cross Ratio (Definition:)

$$(AB, CD) = \frac{\frac{(a-c)}{(b-c)}}{\frac{(a-d)}{(b-d)}} = \frac{(a-c)}{(b-c)} \times \frac{(b-d)}{(a-d)} = \frac{(a-c)(b-d)}{(a-d)(b-c)} = k$$

1. Prove that $(AB, DC) = \frac{1}{k}$

   Well,

$$(AB, DC) = \frac{(a-d)(b-c)}{(a-c)(b-d)} = \frac{1}{\frac{(a-c)(b-d)}{(a-d)(b-c)}} = \frac{1}{k}$$

2. Prove that $(AC, DB) = \frac{1}{1-k}$

   Well,

$$\begin{aligned}
(AC, DB) &= \frac{(a-d)(c-b)}{(a-b)(c-d)} \\
&= \frac{1}{\frac{(a-b)(c-d)}{(a-d)(c-b)}} \\
&= \frac{1}{\frac{(a-b)(c-d)(b-c)}{(a-d)(c-b)(b-c)}} \\
&= \frac{1}{\frac{[ac-ad-bc+bd](b-c)}{(a-d)(c-b)(b-c)}} \\
&= \frac{1}{\frac{(abc-abd-b^2c+b^2d-ac^2+acd+bc^2-bcd)}{(a-d)(c-b)(b-c)}} \\
&= \frac{1}{\frac{(c-b)[-ac-bd+ad+bc]}{(a-d)(c-b)(b-c)}}
\end{aligned}$$

$$= \frac{1}{\frac{(c-b)[(a-d)(b-c)-(a-c)(b-d)]}{(a-d)(c-b)(b-c)}}$$

$$= \frac{1}{\frac{(a-d)(b-c)-(a-c)(b-d)}{(a-d)(b-c)}}$$

$$= \frac{1}{1 - \frac{(a-c)(b-d)}{(a-d)(b-c)}}$$

$$= \frac{1}{1-k}$$

3. Prove that $(AD, BC) = \frac{k-1}{k}$

   Well, using our result from (2), we have:

$$\begin{aligned}
(AD, BC) &= \frac{1}{1-(AC, DB)} \\
&= \frac{1}{1 - \frac{1}{(1-k)}} \\
&= \frac{(1-k)}{(1-k)-1} \\
&= \frac{1-k}{-k} \\
&= \frac{k-1}{k}
\end{aligned}$$

Problem 10

Prove that (for arbitrary O, U) we have:

$$(PQ, OU) \times (QR, OU) = (PR, OU)$$

Solution

Well,

$$\begin{aligned}
(PQ, OU) \times (QR, OU) &= \frac{(p-o)(q-u)}{(p-u)(q-o)} \times \frac{(q-o)(r-u)}{(q-u)(r-o)} \\
&= \frac{(p-o)(q-u)(q-o)(r-u)}{(p-u)(q-o)(q-u)(r-o)} \\
&= \frac{(p-o)(r-u)}{(p-u)(r-o)} \\
&= (PR, OU)
\end{aligned}$$